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Research Article

Solving Some Special Cases of Monomial Ratio Equations Appearing Frequently in Physical and Engineering Problems

Enrique Castillo,[1] Maria Nogal,[2] Jose Antonio Lozano-Galant,[3] and Jose Turmo[4]

[1] *Department of Applied Mathematics and Computational Sciences, University of Cantabria, Avenida de Los Castros s/n, 39005 Santander, Spain*
[2] *Department of Civil, Structural and Environmental Engineering, Trinity College Dublin, Museum Building, College Green, Dublin 2, Ireland*
[3] *Department of Civil Engineering, University of Castilla-La Mancha, Edificio Politécnico, Avenida de Camilo José Cela, 13071 Ciudad Real, Spain*
[4] *Departament d'Enginyeria Civil i Ambiental, Universitat Politècnica de Catalunya, C/ Jordi Girona 1-3, 08021 Barcelona, Spain*

Correspondence should be addressed to Enrique Castillo; castie@unican.es

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We first show that monomial ratio equations are not only very common in Physics and Engineering, but the natural type of equations in many practical problems. More precisely, in the case of models involving scale variables if the used formulas are not of this type they are not physically valid. The consequence is that when estimating the model parameters we are faced with systems of monomial ratio equations that are nonlinear and difficult to solve. In this paper, we provide an original algorithm to obtain the unique solutions of systems of equations made of linear combinations of monomial ratios whose coefficient matrix has a proper null space with low dimension that permits solving the problem in a simple way. Finally, we illustrate the proposed methods by their application to two practical problems from the hydraulic and structural fields.

1. Introduction and Motivation

In this paper, we present a nonlinear problem that commonly appears in Physics and Engineering when dealing with identification problems in general. More precisely, we show how systems of monomial ratio equations are the natural mathematical structure for these problems and present a relevant result that consists in proving that, under mild conditions, no other structure is possible. Finally, we provide a method and an algorithm to solve the problem and give two examples, a water supply and a real structure. To illustrate and for the sake of motivation, we present next a very simple example drawn from the field of structures.

Consider the structure in Figure 1 with the corresponding equilibrium equations based on the stiffness matrix method:

$$
\begin{pmatrix} H_1 \\ V_1 \\ M_1 \\ H_2 \\ V_2 \\ M_2 \\ H_3 \\ V_3 \\ M_3 \\ H_4 \\ V_4 \\ M_4 \end{pmatrix}
=
\begin{pmatrix}
\frac{12E_1I_1}{L_1^3} & 0 & \frac{6E_1I_1}{L_1^2} & \frac{-12E_1I_1}{L_1^3} & 0 & \frac{6E_1I_1}{L_1^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{A_1E_1}{L_1} & 0 & 0 & \frac{-A_1E_1}{L_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{6E_1I_1}{L_1^2} & 0 & \frac{4E_1I_1}{L_1} & \frac{-6E_1I_1}{L_1^2} & 0 & \frac{2E_1I_1}{L_1} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-12E_1I_1}{L_1^3} & 0 & \frac{-6E_1I_1}{L_1^2} & \frac{12E_1I_1}{L_1^3} + \frac{A_2E_2}{L_2} & 0 & \frac{-6E_1I_1}{L_1^2} & \frac{-A_2E_2}{L_2} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-A_1E_1}{L_1} & 0 & 0 & \frac{12E_2I_2}{L_2^3} + \frac{A_1E_1}{L_1} & \frac{6E_2I_2}{L_2^2} & 0 & \frac{-12E_2I_2}{L_2^3} & \frac{6E_2I_2}{L_2^2} & 0 & 0 & 0 \\
\frac{6E_1I_1}{L_1^2} & 0 & \frac{2E_1I_1}{L_1} & \frac{-6E_1I_1}{L_1^2} & \frac{6E_2I_2}{L_2^2} & \frac{4E_1I_1}{L_1} + \frac{4E_2I_2}{L_2} & 0 & \frac{-6E_2I_2}{L_2^2} & \frac{2E_2I_2}{L_2} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{-A_2E_2}{L_2} & 0 & 0 & \frac{A_2E_2}{L_2} + \frac{12E_3I_3}{L_3^3} & 0 & \frac{-6E_3I_3}{L_3^2} & \frac{-12E_3I_3}{L_3^3} & 0 & \frac{-6E_3I_3}{L_3^2} \\
0 & 0 & 0 & 0 & \frac{-12E_2I_2}{L_2^3} & \frac{-6E_2I_2}{L_2^2} & 0 & \frac{12E_2I_2}{L_2^3} + \frac{A_3E_3}{L_3} & \frac{-6E_2I_2}{L_2^2} & 0 & \frac{-A_3E_3}{L_3} & 0 \\
0 & 0 & 0 & 0 & \frac{6E_2I_2}{L_2^2} & \frac{2E_2I_2}{L_2} & \frac{-6E_3I_3}{L_3^2} & \frac{-6E_2I_2}{L_2^2} & \frac{4E_2I_2}{L_2} + \frac{4E_3I_3}{L_3} & \frac{6E_3I_3}{L_3^2} & 0 & \frac{2E_3I_3}{L_3} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{-12E_3I_3}{L_3^3} & 0 & \frac{6E_3I_3}{L_3^2} & \frac{12E_3I_3}{L_3^3} & 0 & \frac{6E_3I_3}{L_3^2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-A_3E_3}{L_3} & 0 & 0 & \frac{A_3E_3}{L_3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{-6E_3I_3}{L_3^2} & 0 & \frac{2E_3I_3}{L_3} & \frac{6E_3I_3}{L_3^2} & 0 & \frac{4E_3I_3}{L_3}
\end{pmatrix}
\begin{pmatrix} h_1 \\ v_1 \\ \theta_1 \\ h_2 \\ v_2 \\ \theta_2 \\ h_3 \\ v_3 \\ \theta_3 \\ h_4 \\ v_4 \\ \theta_4 \end{pmatrix},
\tag{1}
$$

where M_i, H_i, and V_i are the external moments and horizontal and vertical forces applied to node i, θ_i, h_i, and v_i are the rotation and horizontal and vertical displacements of node i, and E_j, I_j, and L_j are the Young modulus, the moment of inertia, and the length of element j, respectively.

Equation (1) can be written in a condensed form as follows:

$$\{f\} = [K]\{\delta\}. \tag{2}$$

One important fact is that the elements in the stiffness matrix K in (1) are sums of monomial ratios and that in many other problems in engineering practice monomial ratios are encountered too. It will be shown that the systems of monomial ratio equations arise in a natural form in Physics and Engineering and in some cases they are the only equations valid and having a physical meaning.

In the design and project of structure stages, the stiffness matrix K, containing the material and geometric properties of the different elements, is assumed to be known, and then the resulting system of equations becomes linear. Contrary, in structural system identification (SSI) problems, the parameters in the stiffness matrix are investigated and then systems of monomial ratio equations (nonlinear) appear.

Frequently, the resulting set of monomial ratio equations have infinitely many solutions when they are initially stated (see [1]). Thus, the engineer must add new constraints in order for the problem or some of its parameters or variables to have a unique solution. If not all the variables are looked for, a unique solution of the system is not needed, but only a unique solution of the subset of variables we are interested in. A variable is observable if it has a unique solution; otherwise, it is unobservable. Identification of the subset of variables to be measured in order to permit another subset of parameters to be calculated is one of the main problems in SSI.

In these cases, a system of equations that are linear relations of monomial ratios is obtained. They can be immediately transformed into a system of polynomial equations by forcing the nonnull character of the variables involved in denominators. However, even if this system is undetermined, a unique solution for a proper subset of variables can be obtained (see [2]). In this context, *a subset of variables is called observable* if the resulting system of equations implies a unique solution for this subset, even though the remaining variables remain undetermined. This leads to the observability or inverse problem (see [3, 4]) that has a relevant role in many engineering problems. In particular, given a system of equations, several questions of interest not only in the mathematical field but in the fields of applied sciences can arise, such as the following.

(1) Is the nonlinear system compatible? That is, has it at least one solution?

(2) If it is compatible, has it a unique or infinitely many solutions?

(3) Can the nonlinear system be solved in some unknowns? That is, is the solution unique for some unknowns?

(4) How many and which subset of variables must be determined to obtain a unique solution for all or a given subset of variables?

Unfortunately, in the existing literature practical methods to directly solve our general system of monomial ratio equations or the associated polynomial systems of equations do not exist. In fact, when trying to solve the systems of polynomial equations resulting from some applications using existing software (Bertini, Mathematica, Maple, etc.), many problems arise because of the huge amount of time required or the impossibility to solve them. Even for systems of small or median size, the computer becomes blocked. The main reasons are mainly due to the following.

(1) Most implemented computer programs try to find all the solutions (real and complex, positive and negative, etc.) and this leads to prohibitive CPU times (in some cases several hours even for small size systems (14 equations)).

(2) When the set of solutions can be restricted (real, nonnegative, etc.), the resulting CPU times are still very large and then not satisfactory.

(3) Since in our examples it is common to have some redundant equations either initially or during the solution process, when there are, even very small errors in the coefficients, there is no solution.

This paper tries to be an invitation and, at the same time, a motivation to people working in the polynomial equation solving field to realize about the important problems that can be solved in the practical fields using some special type of polynomial equations.

Experience with this type of equations leads to the conclusion that tailor made methods are required to solve the systems of polynomial (monomial ratio) equations appearing in applied fields, which must consider (a) most or all unknowns appearing in practical cases being real, (b) the particular character or each unknown (positive, negative, real, etc.), (c) sign variables, and (d) the possibility of identifying the modulus of an unknown but not its sign.

The main originality of this paper goes in these directions; that is, it is addressed to cases of real unknowns and when some unknowns are known to be positive, negative, nonpositive, or nonnegative.

The paper is structured as follows. In Section 2, the observability problem, which leads to many systems of monomial ratio equations, is described. In Section 3, it is shown that systems of monomial ratio equations are of general use in Physics and Engineering. More precisely, it is shown that in some cases they are the only valid ones. In Section 4, the mathematical problem to be solved is clearly stated. In Section 5, a motivating example is given in which the techniques to be implemented in the proposed algorithm are described. In Section 6, an original algorithm to solve this type of problems is presented. In Section 7, a simple water supply problem is discussed. In Section 8, the proposed method is illustrated by its application to some problems of calculus of structures. In Section 9, the proposed method is compared with some other methods used in the existing software packages. Finally, in Section 10, some conclusions are given.

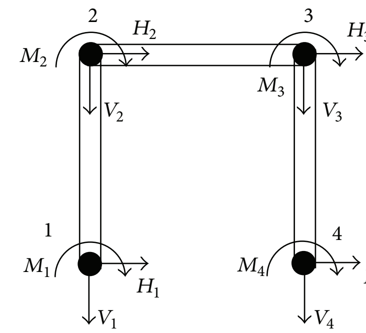

FIGURE 1: A simple structure with the applied forces $H_1, H_2, H_3, H_4, V_1, V_2, V_3$, and V_4 and moments M_1, M_2, M_3, and M_4.

2. The Observability Problem

Observability methods are of special importance in the field of structures where an adequate maintenance and monitoring program can avoid important failures (see, e.g., [5–12]). In this section, the observability problem is introduced. Assume a linear or nonlinear system of equations in a given set of unknowns (variables) with an infinite number of solutions. The observability problem consists of identifying the subset of unknowns that have unique solution. These variables are called *observable* because their unique values can be calculated from the data. The remaining variables are called *unobservable* because they are undetermined.

The observability problem has been dealt with in several areas of research. For example, Castillo et al. [13] have studied the observability problem of linear systems of equations and inequalities in general, some authors [14–17] dealt with the observability of state estimation, and Hue et al. [18] analyzed the observability problem in traffic networks, whereas some other authors [19–21] applied the observability methods to the structural identification.

In applied sciences, it is common to have systems of equations, written in terms of matrix equations, that have unknowns in both sides. In such cases, it is important to join together the unknowns on one side of the equation and the known values on the coefficient and independent term matrices. For illustrative purposes, assume that we have the system of equations

$$\{f\} = \begin{Bmatrix} f_0^{p\times 1} \\ f_1^{q\times 1} \end{Bmatrix} = [K]\{\delta\} = \begin{pmatrix} K_{00}^{p\times r} & K_{01}^{p\times s} \\ K_{10}^{q\times r} & K_{11}^{q\times s} \end{pmatrix} \begin{Bmatrix} \delta_0^{r\times 1} \\ \delta_1^{s\times 1} \end{Bmatrix}, \quad (3)$$

where $K_{00}^{p\times r}$, $K_{01}^{p\times s}$, $K_{10}^{q\times r}$, and $K_{11}^{q\times s}$ are the partitioned matrices of $[K]$ with their dimensions given as superindices and $\delta_0^{r\times 1}$, $\delta_1^{s\times 1}$, $f_1^{p\times 1}$, and $f_0^{q\times 1}$ are the partitioned matrices of $\{\delta\}$ and $\{f\}$, respectively, and the subindices 0 and 1 refer to unknown and known variables, respectively.

System (3) can be rearranged in an equivalent form so that the unknown variables appear in the left hand side and the known variables in the right hand side, as follows:

$$\begin{aligned}[B]\{z\} &= \begin{pmatrix} K_{10}^{q\times r} & 0^{q\times p} \\ K_{00}^{p\times r} & -I^{p\times p} \end{pmatrix} \begin{Bmatrix} \delta_0^{r\times 1} \\ f_0^{p\times 1} \end{Bmatrix} \\ &= \begin{Bmatrix} f_1^{q\times 1} - K_{11}^{q\times s} \times \delta_1^{s\times 1} \\ -K_{01}^{p\times s} \times \delta_1^{s\times 1} \end{Bmatrix} = \{D\},\end{aligned} \tag{4}$$

where $0^{q\times p}$ and $I^{p\times p}$ are the null and the identity matrices of the indicated dimensions, respectively, and $\{D\}$ is a column matrix of known elements.

In general, system (4) need not to be compatible (have a solution). In fact, matrix $\{D\}$ must satisfy some conditions for system (4) to have a solution. In order to check if the system has a solution, it is sufficient to calculate the null space $[V]$ of $[B]^T$ and check that $[V][D] = \{0\}$. If this holds, the system is compatible; otherwise, it has no solution (see [2]).

The general solution (the set of all solutions) of system (4) has the following structure (see [2]):

$$\{z\} = \{z_0\} + [W]\{\rho\}, \tag{5}$$

where $\{z_0\}$ is a particular solution of system (4) and $[W]\{\rho\}$ is the set of all solutions of the associated homogeneous system of equations (a linear space of solutions, where the columns of $[W]$ are a basis of the null space of B and the elements of the column matrix $\{\rho\}$ are arbitrary real values which represent the coefficients of all possible linear combinations).

It is interesting to note that a variable has unique solution not only when matrix $[W]$ has zero dimensions (it does not exist), but when the associated row in matrix $[W]$ is null. Thus, the examination of matrix $[W]$ and the identification of its null rows lead to the identification of the observable variables (subset of variables with a unique solution).

To obtain matrix $[W]$, we need to obtain the null space of matrix $[B]$, which can be done with the help of standard subroutines or functions provided by well known packages, such as MATLAB or Mathematica.

Unfortunately, not all systems arising in engineering problems are linear as (4), and then alternative methods must be used. In this paper, we face a nonlinear problem and present an original method to solve the observability problem. Fortunately, we can work with a linear system of monomial ratios in the variables, which allows us to use the linear case tools and methods for these variables.

3. Formulas Appearing in Physics and Engineering

In this section, it is demonstrated that monomial ratio formulas are not a coincidence but some condition to be satisfied in order to have physically valid formulas.

Some very well known examples of physically valid formulas are the following:

$$\begin{aligned} f &= ma; \\ e &= \frac{at^2}{2}; \\ K &= \frac{mv^2}{2}; \\ U_G &= -\frac{Gm_1m_2}{r^2}; \\ F &= \frac{q_1q_2}{4\pi\varepsilon_0 r^2}. \end{aligned} \tag{6}$$

As it will be seen all these formulas must be monomial ratios, which in some cases can degenerate to a monomial (denominator equal to one).

In a physical system, there are some fundamental variables, such as length, time, and space; from them, secondary or derived variables are obtained by certain, more or less complicated, formulas. In other cases, formulas relate different variables, not necessarily fundamental. However, not every formula generates a valid variable, but only those satisfying some extra conditions (see [22, pages 35–70]). For formula $u = u(x_1, x_2, \ldots, x_n)$ to be physically valid, it is necessary that a change of location or/and scale of the independent variables $x_1, x_2, \ldots, x_n$ produce a change of location or/and scale of the derived or dependent variable u. In other words, the formula must satisfy the following functional equation:

$$\begin{aligned} &u(k_1x_1 + t_1, k_2x_2 + t_2, \ldots, k_nx_n + t_n) \\ &\quad = K(k_1, k_2, \ldots, k_n; t_1, t_2, \ldots, t_n)\, u(x_1, x_2, \ldots, x_n) \\ &\qquad + T(k_1, k_2, \ldots, k_n; t_1, t_2, \ldots, t_n) \\ &\qquad\qquad (k_i, x_i \in \mathbb{R}_{++};\ i = 1, 2, \ldots, n), \end{aligned} \tag{7}$$

where $u(x_1, x_2, \ldots, x_n)$ is the formula that gives the derived variable as a function of the fundamental variables $x_1, x_2, \ldots, x_n$, k_i is the factor used to change units in variable i, t_i is the location change of variables i, and T and K are functions associated with the location and scale changes of the derived variable, respectively. In other words, if formula $u(x_1, x_2, \ldots, x_n)$ does not satisfy (7), it is not physically valid.

When a variable is allowed for location and scale changes, we say that it has an *interval scale*. If a variable is allowed for scale changes only, we say that it has a *ratio scale*. Examples of variables with interval scales are time, temperature, and location. Examples of ratio scale variables are length, area, volume, speed, and acceleration.

The following theorem and corollary due to [22] provide the only valid formulas for some physical formulas.

Theorem 1 (general formula for interval scale variables). *The general form of dependent real-valued variables with interval scale nonconstant and continuous-at-a-point when all* *fundamental or independent variables have ratio scale, that is, the general solutions of the functional equation*

$$\begin{aligned} & u\left(k_1 x_1, k_2 x_2, \ldots, k_n x_n\right) \\ & \quad = K\left(k_1, k_2, \ldots, k_n\right) u\left(x_1, x_2, \ldots, x_n\right) \\ & \qquad + T\left(k_1, k_2, \ldots, k_n\right); \\ & \qquad\qquad \left(k_1, k_2, \ldots, k_n\right), \left(x_1, x_2, \ldots, x_n\right) \in \mathbb{R}_{++}^n, \end{aligned} \tag{8}$$

where $(x_1, x_2, \ldots, x_n)$ *is the vector of the independent variables,* $(k_1, k_2, \ldots, k_n)$ *is the vector of scale changes for all variables,* $\mathbb{R}_{++}$ *is the set of positive real numbers, and* T *and* K*, as before, are the functions associated with the location and scale changes of the derived variable, respectively, are*

$$\begin{aligned} u\left(x_1, x_2, \ldots, x_n\right) &= \sum_{i=1}^{n} c_i \log x_i + b, \\ K\left(k_1, k_2, \ldots, k_n\right) &= 1, \\ T\left(k_1, k_2, \ldots, k_n\right) &= \sum_{i=1}^{n} c_i \log k_i, \end{aligned} \tag{9}$$

$$\begin{aligned} u\left(x_1, x_2, \ldots, x_n\right) &= a \prod_{i=1}^{n} x_i^{c_i} + b, \\ K\left(k_1, k_2, \ldots, k_n\right) &= \prod_{i=1}^{n} k_i^{c_i}, \\ T\left(k_1, k_2, \ldots, k_n\right) &= b\left[1 - \prod_{i=1}^{n} k_i^{c_i}\right], \end{aligned} \tag{10}$$

where c_i*,* $i = 1, 2, \ldots, n$*,* a*, and* b *are arbitrary real constants.*

Corollary 2 (ratio scale variables). *The general form of dependent real-valued variables with ratio scale nonconstant and continuous-at-a-point when all fundamental or independent variables have ratio scale, that is, the general solution of the functional equation*

$$\begin{aligned} & u\left(k_1 x_1, k_2 x_2, \ldots, k_n x_n\right) \\ & \quad = K\left(k_1, k_2, \ldots, k_n\right) u\left(x_1, x_2, \ldots, x_n\right); \\ & \qquad\qquad \left(k_1, k_2, \ldots, k_n\right), \left(x_1, x_2, \ldots, x_n\right) \in \mathbb{R}_{++}^n, \end{aligned} \tag{11}$$

is

$$\begin{aligned} u\left(x_1, x_2, \ldots, x_n\right) &= a \prod_{i=1}^{n} x_i^{c_i}; \\ K(\mathbf{k}) &= \prod_{i=1}^{n} k_i^{c_i}, \end{aligned} \tag{12}$$

with $a \neq 0$ *and* $\sum_{i=1}^{n} c_i^2 \neq 0$*.*

Some interesting references on functional equations are [23, 24]. The proofs of Theorem 1 and Corollary 2 can be seen in [25, 26] or [27].

Since the expression $a\prod_{i=1}^{n} x_i^{c_i}$ can be written as $a\prod_{c_i>0} x_i^{c_i} / \prod_{c_i<0} x_i^{|c_i|}$, it is a monomial ratio. This means that physical formulas which are not of the logarithmic type in (9) all involve monomial ratios. We can add that they are the most common.

4. Statement of the Problem

In this section, the type of systems of nonlinear equations to deal with in this paper is introduced and justified. Motivated by the monomial ratio structure of physically valid formulas, a system of nonlinear equations of the following form is assumed:

$$\sum_{j=1}^{n} a_{ij} \prod_{s=1}^{p} x_s^{\alpha_{sj}} = b_i; \quad i = 1, 2, \ldots, m. \tag{13}$$

That is, each equation is assumed to be a linear combination of multidimensional monomial ratios in the variables x_s, $s = 1, 2, \ldots, p$, where $a_{ij}, b_i \in \mathbb{R}$ are real coefficients and $\prod_{s=1}^{p} x_s^{\alpha_{sj}}$ are monomial ratios, where the positive and negative α_{sj} values give the numerator and the denominator monomials, respectively.

The aim of this paper consists of solving the observability problem in system (13). More precisely, the variables in the set $\mathcal{X} \equiv \{x_1, x_2, \ldots, x_n\}$ which are observable (can be determined) when the variables in a subset $\mathcal{X}_1 \subset \mathcal{X}$ are known are investigated. Let $\mathcal{S} \equiv \{s \mid x_s \in \mathcal{X}_1\}$, that is, the set of indices of known variables, that is, the variables x in set $\mathcal{X}_1$.

To this end, we write system (13) in the following equivalent form:

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} \prod_{s=1}^{p} x_s^{\alpha_{sj}} = \sum_{j}^{n} a_{ij} \prod_{s \in \mathcal{S}}^{p} x_s^{\alpha_{sj}} \prod_{s \notin \mathcal{S}}^{p} x_s^{\alpha_{sj}} = \sum_{j}^{n} a_{ij}^* \prod_{s \notin \mathcal{S}}^{p} x_s^{\alpha_{sj}} = b_i; \\ i = 1, 2, \ldots, m, \end{aligned} \tag{14}$$

where

$$a_{ij}^* = a_{ij} \prod_{s \in \mathcal{S}}^{p} x_s^{\alpha_{sj}}; \quad i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n. \tag{15}$$

Since the known variables are the only ones that appear in the elements a_{ij}^* of matrix A^*, this matrix is a constant matrix, and we can interpret system (14) as a system of m linear equations in the n unknown monomial ratios $v_j = \prod_{s \notin \mathcal{S}}^{p} x_s^{\alpha_{sj}}$, $j = 1, 2, \ldots, n$.

It is well known that linear systems of equations can have no solution, one solution, or infinitely many solutions. Thus, it is important to know whether or not this system has solution, and it is more important to identify the set of all its solutions. The null space of matrix A^*, with elements $\{a_{ij}^*\}$, allows us to write the set of all solutions of (14) as

$$\prod_{s \notin \mathcal{S}}^{p} x_s^{\alpha_{sj}} = v_j^0 + W\rho; \quad j = 1, 2, \ldots, n, \tag{16}$$

where v_j^0 is any particular solution of system (14), W is the null space matrix of A^*, such that $A^*W = \mathbf{0}$ with $\mathbf{0}$ the null matrix, A_j^* is the row j of matrix W, and ρ is a column matrix of arbitrary real numbers.

Consequently, the subset of observable products corresponds to the null rows of the null space matrix W of matrix A^*.

Let v_j^0, $j \in \mathcal{O}$, be the set of observable products. Then, we have

$$\prod_{s \notin \mathcal{S}}^{p} x_s^{\alpha_{js}} = \beta_j v_j^0; \quad j \in \mathcal{O}, \tag{17}$$

where we assume $|v_j^0| > 0$, β_j is 1 or -1 in order to make $\beta_j v_j^0$ positive. Taking logarithms leads to the system of linear equations

$$\sum_{s \notin \mathcal{S}}^{p} \alpha_{js} \log x_s = \log\left(\beta_j v_j^0\right); \quad j \in \mathcal{O}. \tag{18}$$

However, for the unobservable products, we have

$$\prod_{s \notin \mathcal{S}}^{p} x_s^{\alpha_{js}} = \beta_j v_j^0; \quad j \notin \mathcal{O}, \tag{19}$$

and taking logarithms we get

$$\sum_{s \notin \mathcal{S}}^{p} \alpha_{js} \log x_s = \log\left(\beta_j v_j^0\right); \quad j \notin \mathcal{O}. \tag{20}$$

If for some $j \notin \mathcal{O}$ the expression in (20) is a linear combination of expressions in the left hand side of (18), then the associated value $\log v_j$ can be calculated by the same linear combination of $\log(\beta_j v_j^0)$, and then the product $\prod_{s \notin \mathcal{S}}^{p} x_s^{\alpha_{js}}$ becomes observable too.

In order to check that the linear combination exists for the product $\prod_{s \notin \mathcal{S}}^{p} x_s^{\alpha_{js}}$, first a set of generators $\mathcal{W}$ of the subspace orthogonal to the linear subspace generated by the set of vectors

$$\mathcal{V} \equiv \left\{\boldsymbol{\alpha}_j = \left(\alpha_{j1}, \alpha_{j1}, \ldots, \alpha_{jn}\right);\ j \in \mathcal{O}\right\} \tag{21}$$

is obtained and whether or not the vector $\boldsymbol{\alpha}_j$ is orthogonal to every vector in $\mathcal{W}$ is checked. In such a case, the product $\prod_{s \notin \mathcal{S}}^{p} x_s^{\alpha_{js}}$ becomes observable.

This methodology supplies additional observable products that were not included in the first list of observable products. Then, the associated information is stored in the knowledge base K and the process repeated until no more variables can be observed.

5. Motivating Example

In this section, a particular example of some type of polynomial equations that appear frequently in physical and engineering problems is introduced. Next, the example is solved, step by step, by applying some of the techniques to be developed later. In this process, the tools and techniques used to solve these systems of equations are illustrated to motivate the algorithm proposed at the end of the paper.

Consider the following polynomial system of twelve equations in sixteen unknown variables (x_1 to x_{16}):

$$\begin{aligned}
-6 &= -\frac{3}{8}x_6^2x_8x_9^2x_{10}^2 + x_7x_5^3x_6 + \frac{1}{2}x_2^2x_5^2x_9x_{10}, \\
\frac{77}{2} &= \frac{5}{8}x_1x_2^2x_{14}^2x_{16}^2 - \frac{1}{2}x_2x_3^2x_{15}^3x_{16}^2 + \frac{5}{8}x_1x_2^2x_{15}^{-2}, \\
\frac{15}{4} &= \frac{3}{16}x_6^2x_8x_9^2x_{10}^2 - \frac{3}{8}x_7x_5^3x_6 - \frac{3}{8}x_2^2x_5^2x_9x_{10}v \\
&\quad - \frac{3}{8}x_1x_2x_3x_{14}^2 + \frac{3}{8}x_1x_2^{-1}x_4^2x_6 \\
&\quad + \frac{3}{8}x_2x_4^2x_5^2x_{15}, \\
-1 &= -\frac{3}{8}x_6^2x_8x_9^2x_{10}^2 + \frac{1}{2}x_7x_5^3x_6 + x_2^2x_5^2x_9x_{10} \\
&\quad - \frac{3}{4}x_1x_2x_3x_{14}^2 + x_1x_2^{-1}x_4^2x_6 \\
&\quad + \frac{1}{2}x_2x_4^2x_5^2x_{15}, \\
18 &= \frac{9}{4}x_2x_3^2x_{15}^3x_{16}^2, \\
12 &= \frac{15}{16}x_1x_2x_3x_{14}^2 - \frac{3}{8}x_1x_2^{-1}x_4^2x_6 + \frac{3}{8}x_1x_3^2x_4x_6^3, \\
29 &= \frac{3}{8}x_1x_2x_3x_{14}^2 + \frac{1}{2}x_1x_2^{-1}x_4^2x_6 + 2x_2x_4^2x_5^2x_{15} \\
&\quad + \frac{1}{2}x_1x_3^2x_4x_6^3, \\
29 &= \frac{9}{8}x_1x_2x_3x_{14}^2 + \frac{1}{2}x_2x_4^2x_5^2x_{15} + x_1x_3^2x_4x_6^3, \\
-\frac{87}{2} &= -\frac{5}{8}x_1x_2^2x_{14}^2x_{16}^2 - \frac{5}{8}x_1x_2^2x_{15}^{-2} - x_1x_5^2x_7, \\
-\frac{17}{4} &= -\frac{3}{16}x_6^2x_8x_9^2x_{10}^2 + \frac{3}{8}x_7x_5^3x_6 + \frac{3}{8}x_2^2x_5^2x_9x_{10} \\
&\quad - x_{11}x_{12}x_{13}, \\
-18 &= -\frac{7}{4}x_2x_3^2x_{15}^3x_{16}^2 - x_3x_{11}^{-1}x_{12}^2, \\
-\frac{31}{2} &= -\frac{9}{16}x_1x_2x_3x_{14}^2 - \frac{3}{8}x_2x_4^2x_5^2x_{15} - \frac{3}{8}x_1x_3^2x_4x_6^3 \\
&\quad - x_4^2x_{11}x_{12}^{-1}x_{13},
\end{aligned} \tag{22}$$

which written in matrix form becomes as follows:

$$\begin{pmatrix} -6 \\ \dfrac{77}{2} \\ \dfrac{15}{4} \\ -1 \\ 18 \\ 12 \\ 29 \\ 29 \\ -\dfrac{87}{2} \\ -\dfrac{17}{4} \\ -18 \\ -\dfrac{31}{2} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{3}{8} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{5}{8} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{5}{8} & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{16} & -\frac{3}{8} & -\frac{3}{8} & 0 & -\frac{3}{8} & \frac{3}{8} & \frac{3}{8} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{8} & \frac{1}{2} & 1 & 0 & -\frac{3}{4} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{9}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{15}{16} & -\frac{3}{8} & 0 & \frac{3}{8} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{8} & \frac{1}{2} & 2 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{9}{8} & 0 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{5}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5}{8} & -1 & 0 & 0 & 0 \\ 0 & -\frac{3}{16} & \frac{3}{8} & \frac{3}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{7}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{9}{16} & 0 & -\frac{3}{8} & -\frac{3}{8} & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1x_2^2x_{14}^2x_{16}^2 \\ x_6^2x_8x_9^2x_{10}^2 \\ x_7x_5^3x_6 \\ x_2^2x_5^2x_9x_{10} \\ x_2x_3^2x_{15}^3x_{16}^2 \\ x_1x_2x_3x_{14}^2 \\ x_1x_2^{-1}x_4^2x_6 \\ x_2x_4^2x_5^2x_{15} \\ x_1x_3^2x_4x_6^3 \\ x_1x_2^2x_{15}^{-2} \\ x_1x_5^2x_7 \\ x_{11}x_{12}x_{13} \\ x_3x_{11}^{-1}x_{12}^2 \\ x_4^2x_{11}x_{12}^{-1}x_{13} \end{pmatrix}. \quad (23)$$

The idea consists in writing all unknowns in the right column matrix denoted by z, so that the rectangular matrix of coefficients B and the left hand column matrix, denoted by D, become constant.

To reduce the set of solutions or even to get a unique solution, the engineer observes the values of some variables. To illustrate, in this example, it is assumed that this information (the result of such observations) comes in the form

$$\begin{aligned} x_{14} &= x_{16} = 2; \\ x_{15} &= 1. \end{aligned} \quad (24)$$

If in (23) the values in (24) are replaced, the new system of equations is obtained as follows:

$$\begin{pmatrix} -6 \\ \dfrac{77}{2} \\ \dfrac{15}{4} \\ -1 \\ 18 \\ 12 \\ 29 \\ 29 \\ -\dfrac{87}{2} \\ -\dfrac{17}{4} \\ -18 \\ -\dfrac{31}{2} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{3}{8} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & \frac{5}{8} & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{16} & -\frac{3}{8} & -\frac{3}{8} & 0 & -\frac{3}{2} & \frac{3}{8} & \frac{3}{8} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{8} & \frac{1}{2} & 1 & 0 & -3 & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{15}{4} & -\frac{3}{8} & 0 & \frac{3}{8} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{2} & \frac{1}{2} & 2 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{9}{2} & 0 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5}{8} & -1 & 0 & 0 & 0 \\ 0 & -\frac{3}{16} & \frac{3}{8} & \frac{3}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{9}{4} & 0 & -\frac{3}{8} & -\frac{3}{8} & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_2^2x_1 \\ x_6^2x_8x_9^2x_{10}^2 \\ x_7x_5^3x_6 \\ x_2^2x_5^2x_9x_{10} \\ x_3^2x_2 \\ x_3x_2x_1 \\ x_1x_2^{-1}x_4^2x_6 \\ x_5^2x_4^2x_2 \\ x_1x_3^2x_4x_6^3 \\ x_2^2x_1 \\ x_1x_5^2x_7 \\ x_{11}x_{12}x_{13} \\ x_3x_{11}^{-1}x_{12}^2 \\ x_4^2x_{11}x_{12}^{-1}x_{13} \end{pmatrix}, \quad (25)$$

in which a new column matrix of monomials is obtained, where x_{14}, x_{15}, and x_{16} have disappeared, and a modified constant coefficient matrix B arises in which the corresponding columns $(1, 5, 6$, and $8)$ have been multiplied by constants $16 = x_{14}^2 x_{16}^2$, $4 = x_{15}^3 x_{16}^2$, $4 = x_{14}^2$, and $1 = x_{15}$, respectively, and all other columns of matrix B and the whole matrix D remain unchanged.

Observing the monomials in the right column matrix z in (25) shows that a duplicated monomial $(x_2^2 x_1)$ exists in rows 1 and 10. This duplicate monomial can be avoided by adding columns 1 and 10 of the coefficient matrix B in (25) and then the new modified simpler system of equations is obtained as follows:

$$\begin{pmatrix} -6 \\ \frac{77}{2} \\ \frac{15}{4} \\ -1 \\ 18 \\ 12 \\ 29 \\ 29 \\ -\frac{87}{2} \\ -\frac{17}{4} \\ -18 \\ -\frac{31}{2} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{3}{8} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{85}{8} & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{16} & -\frac{3}{8} & -\frac{3}{8} & 0 & -\frac{3}{2} & \frac{3}{8} & \frac{3}{8} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{8} & \frac{1}{2} & 1 & 0 & -3 & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{15}{4} & -\frac{3}{8} & 0 & \frac{3}{8} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{2} & \frac{1}{2} & 2 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{9}{2} & 0 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\ -\frac{85}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -\frac{3}{16} & \frac{3}{8} & \frac{3}{8} & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -7 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{9}{4} & 0 & -\frac{3}{8} & -\frac{3}{8} & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_2^2 x_1 \\ x_6^2 x_8 x_9^2 x_{10}^2 \\ x_7 x_5^3 x_6 \\ x_2^2 x_5^2 x_9 x_{10} \\ x_3^2 x_2 \\ x_3 x_2 x_1 \\ x_1 x_2^{-1} x_4^2 x_6 \\ x_5^2 x_4^2 x_2 \\ x_1 x_3^2 x_4 x_6^3 \\ x_1 x_5^2 x_7 \\ x_{11} x_{12} x_{13} \\ x_3 x_{11}^{-1} x_{12}^2 \\ x_4^2 x_{11} x_{12}^{-1} x_{13} \end{pmatrix}, \quad (26)$$

whose associated coefficient matrix B has dimensions (12×13) instead of the dimensions (12×14) of this matrix in system (25). The new matrix z has dimensions (13×1).

System (26) is linear in the 13 unknown monomials. Thus, it can be solved by obtaining a particular solution and the null space (this null space is the orthogonal subspace to the subspace generated by the rows of matrix B in (26)) or using the Gauss-Jordan reduction. This general solution is (see [2])

$$\begin{pmatrix} x_2^2 x_1 \\ x_6^2 x_8 x_9^2 x_{10}^2 \\ x_7 x_5^3 x_6 \\ x_3^2 x_2 \\ x_3 x_2 x_1 \\ x_1 x_2^{-1} x_4^2 x_6 \\ x_5^2 x_4^2 x_2 \\ x_1 x_3^2 x_4 x_6^3 \\ x_1 x_5^2 x_7 \\ x_{11} x_{12} x_{13} \\ x_3 x_{11}^{-1} x_{12}^2 \\ x_4^2 x_{11} x_{12}^{-1} x_{13} \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -6 \\ 2 \\ 2 \\ 4 \\ 8 \\ 16 \\ 1 \\ 2 \\ 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \left(x_2^2 x_5^2 x_9 x_{10} \right), \quad (27)$$

where the first matrix in the right hand side is a particular solution and the second matrix, denoted by N, contains in its columns the generators (in this case only one) of the null space.

Equation (27) gives the general solution of system (26), where the monomials with a unique solution correspond to the null rows of N; that is,

$$\begin{pmatrix} x_2^2 x_1 \\ x_3^2 x_2 \\ x_3 x_2 x_1 \\ x_1 x_2^{-1} x_4^2 x_6 \\ x_5^2 x_4^2 x_2 \\ x_1 x_3^2 x_4 x_6^3 \\ x_1 x_5^2 x_7 \\ x_{11} x_{12} x_{13} \\ x_3 x_{11}^{-1} x_{12}^2 \\ x_4^2 x_{11} x_{12}^{-1} x_{13} \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 2 \\ 4 \\ 8 \\ 16 \\ 1 \\ 2 \\ 4 \\ 2 \end{pmatrix}. \tag{28}$$

Next, taking logarithms in system (28) we get

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}$$

$$\cdot \log \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \end{pmatrix} = \log \begin{pmatrix} 4 \\ 2 \\ 2 \\ 4 \\ 8 \\ 16 \\ 1 \\ 2 \\ 4 \\ 2 \end{pmatrix} \tag{29}$$

and taking logarithms of the multiple solution variables the following expression is obtained:

$$\log \begin{pmatrix} x_6^2 x_8 x_9^2 x_{10}^2 \\ x_7 x_5^3 x_6 \\ x_2^2 x_5^2 x_9 x_{10} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\cdot \log \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13} \end{pmatrix}. \tag{30}$$

Solving system (29), it can be concluded that all 10 variables have a unique solution; that is,

$$\begin{aligned} x_1 &= 1; \\ x_2 &= 2; \\ x_3 &= 1; \\ x_4 &= 2; \\ x_5 &= 1; \\ x_6 &= 2; \\ x_7 &= 1; \\ x_{11} &= 1; \\ x_{12} &= 2; \\ x_{13} &= 1. \end{aligned} \tag{31}$$

In (27), the following subset of variables with multiple solution was obtained; that is, with indeterminate values,

$$\begin{pmatrix} x_6^2 x_8 x_9^2 x_{10}^2 \\ x_7 x_5^3 x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} \left(x_2^2 x_5^2 x_9 x_{10} \right). \tag{32}$$

Now, it can be checked whether or not this subset of variables can be calculated from the subset of observable variables in (31). Though in this example it is obvious that $x_7x_5^3x_6$ can be calculated because its three single factors are known (see (31)), in general this can be done by multiplying the matrices in (30) and (31) to get

$$\begin{pmatrix} 0&0&0&0&0&2&0&1&2&2&0&0&0 \\ 0&0&0&0&3&1&1&0&0&0&0&0&0 \\ 0&2&0&0&2&0&0&0&1&1&0&0&0 \end{pmatrix} \begin{pmatrix} 0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\\1&0&0\\0&1&0\\0&0&1\\0&0&0\\0&0&0\\0&0&0 \end{pmatrix} = \begin{pmatrix} 1&2&2\\0&0&0\\0&1&1 \end{pmatrix} \tag{33}$$

and then locating the null rows of the resulting matrix in (33), which in our case is the second row; that is, $x_7x_5^3x_6$ can be observed. The value of $x_7x_5^3x_6$ can be calculated using the second row of the left matrix in (30) and the particular solution in (31); that is,

$$\begin{aligned} &\log\left(x_7x_5^3x_6\right) \\ &= \begin{pmatrix} 0&0&0&0&3&1&1&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} 0.000\\0.693\\0.000\\0.693\\0.000\\0.693\\0.000\\0.000\\0.000\\0.000\\0.000\\0.693\\0.000 \end{pmatrix} \\ &= 0.693 \Longleftrightarrow x_7x_5^3x_6 = 2. \end{aligned} \tag{34}$$

The values of the variables in (28) can be stored in a knowledge base for later use.

Once it is known that $x_7x_5^3x_6 = 2$ going to (32) it results in the following:

$$\begin{pmatrix} x_6^2x_8x_9^2x_{10}^2 \\ x_2^2x_5^2x_9x_{10} \end{pmatrix} = \begin{pmatrix} 32 \\ 8 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} x_8x_9^2x_{10}^2 \\ x_9x_{10} \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix}, \tag{35}$$

and taking logarithms we obtain a system with unique solution for x_8

$$\begin{aligned} \log\begin{pmatrix} 8 \\ 2 \end{pmatrix} &= \begin{pmatrix} 1&2&2 \\ 0&1&1 \end{pmatrix} \log\begin{pmatrix} x_8 \\ x_9 \\ x_{10} \end{pmatrix}, \\ \log\begin{pmatrix} x_8 \\ x_9 \end{pmatrix} &= \begin{pmatrix} 0.693 \\ 0.693 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \log\left(x_{10}\right), \\ \begin{pmatrix} x_8 \\ x_9x_{10} \end{pmatrix} &= \begin{pmatrix} 2 \\ 2 \end{pmatrix}. \end{aligned} \tag{36}$$

Finally, the desired solution of system (23) results in

$$\begin{aligned} x_1 &= 1, \\ x_2 &= 2, \\ x_3 &= 1, \\ x_4 &= 2, \\ x_5 &= 1, \\ x_6 &= 2, \\ x_7 &= 1, \\ x_8 &= 2, \\ x_9x_{10} &= 2, \\ x_{11} &= 1, \\ x_{12} &= 2, \\ x_{13} &= 1, \\ x_{14} &= 2, \\ x_{15} &= 1, \\ x_{16} &= 2. \end{aligned} \tag{37}$$

Note that if x_9 or x_{10} is given, the solution is unique; otherwise, there are infinitely many solutions for x_9 and x_{10}, but not for the rest of variables (they have a unique solution).

Once this example has been introduced, the proposed algorithm is described in the next section.

6. Proposed Algorithm

A detailed analysis of the previous example shows that it is equivalent to an iterative application of three operations.

Operation 1. It consists of transforming the equations by linear operations (separating known and unknown variables).

Operation 2. It consists of reducing some equations by the other ones (unknowns with unique solutions are identified).

Operation 3. It consists of factoring out a variable from an equation that is divisible by this variable. This operation is implied by making linear algebra on the logarithms of the equations.

The following algorithm summarizes the ideas explained above and explains in detail the steps to be followed to implement the proposed method. With this algorithm, the product variables which are observable can be determined.

The problem we are dealing with is a special one in which we have the following elements.

(1) Problems with a graph structure made of nodes and arcs are considered, as well as the equations in the system mean equilibrium equations. So, the corresponding matrix structure is banded; that is, the number of monomials involved in a given equation cannot be high because the number of links (elements) joining at a node is only a few compared with the total number of arcs. This structure is very frequent in engineering (water supply problems, traffic networks, calculus of structures, finite elements, etc.).

(2) Operation 3 normally occurs, so it is a very important step in the proposed method. In addition, the values of single variables can also be determined in Operation 2, because the learnt values of the variables identified in one recursive step are replaced into the monomials for the following step. In other words, it can happen that, having no Operation 3 in one step, we can continue with the following iteration because the information learnt in Operation 2 can consist of single variables values; that is, the lack of occurrence of Operation 3 in a given iteration does not necessarily imply the end of the process.

(3) All considered variables in the systems of equations are assumed to be real and some are known to be positive, such as stiffness, areas, and diameters, so that the binary trees resulting from the two sign possibilities disappear and make the problem solving much simpler. All these considerations indicate that an ad hoc method adjusted to this special case of polynomial equations is required. In other words, the use of a general method leads to a much larger CPU time, as it has been demonstrated in the paper by a comparison.

(4) Unrestricted variables have been represented as a product of a positive variable (modulus) and a sign variable. This simplifies the complexity of the problem because we avoid sign discussions and associated trees.

(5) The case of zero values of the variables, apart from the boundary conditions and loads, that are data, occurs only by coincidence (with zero probability).

All these considerations have led to the following algorithm, which is illustrated in Figure 2, where for the sake of illustrations the step numbers have been indicated.

6.1. Proposed Algorithm

Input. It is matrices K, δ, and f of the system of nonlinear equations $f = K\delta$ being considered, the subset $\overline{V}_0 \subset V$ of observed (known) variables, and their respective values $\overline{V}_0^0$.

Output. It is the subset $\mathcal{O}$ of observable variables and their values $\mathcal{O}^0$ based on $\overline{V}_0$ and $\overline{V}_0^0$ of observed variables that satisfy the system of equations $f = K\delta$. In addition, the updated matrices B and D are returned.

Step 1 (build the initial system of equations). From the system $f = K\delta$ in (3) build the system $Bz = D$ in (4), where B is a constant coefficient matrix, the unknown variables are grouped into z, and the known data are grouped into D.

Step 2 (check whether or not known variables are available). If there are no new known variables, stop the process and return the list of observable variables $\mathcal{O}$. Otherwise, go to Step 3.

Step 3 (replace known variables by their values and modify the system of equations). Replace all known variables in matrix z by their values and move them to matrix B.

Step 4 (test for known product variables). Check if there are new known product variables. If there are, subtract the corresponding column from the independent term matrix D, and remove the column from B and the variable from z.

Step 5 (check for replicated product variables). Test whether or not replicated product variables exist in matrix z. If there exist such variables, remove them by merging (adding the columns) the corresponding columns in B and merging the corresponding rows in z.

Step 6 (solve the system of linear equations in product variables). Solve the system of equations obtaining its general solution (all solutions).

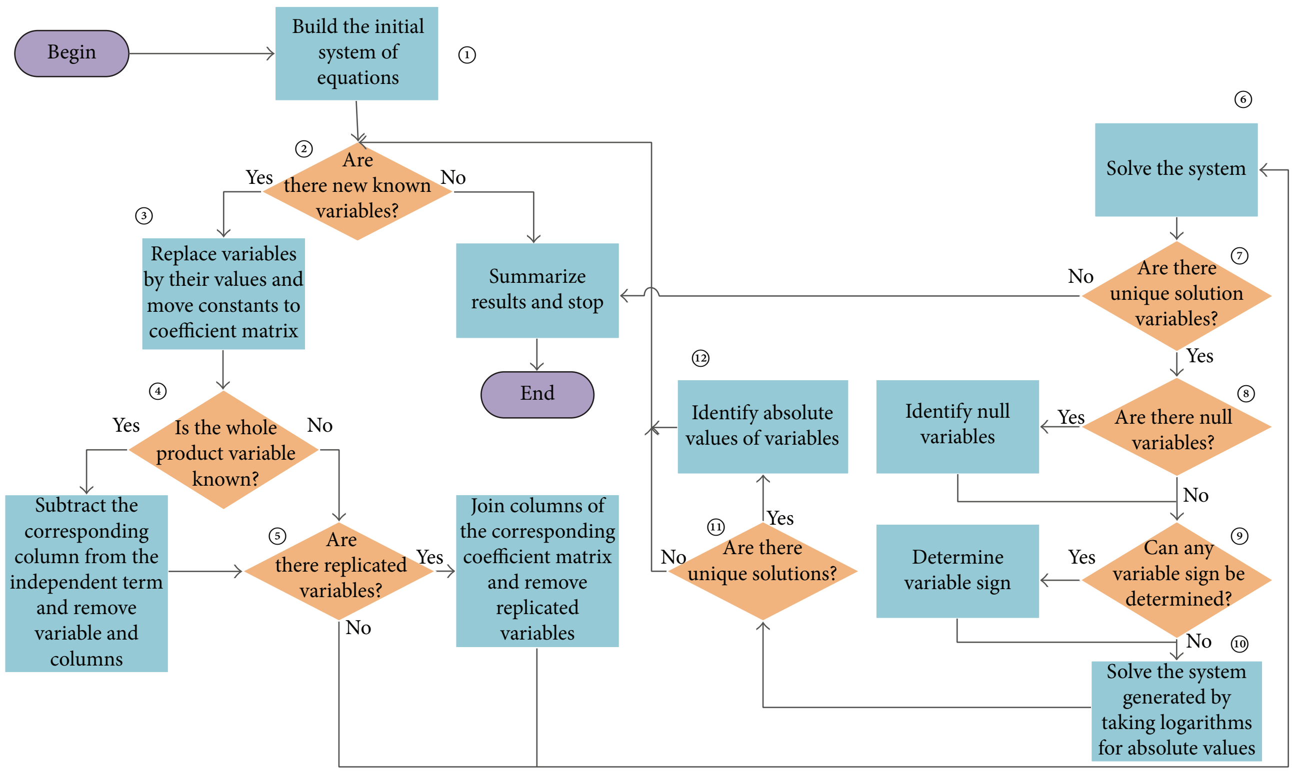

Figure 2: Algorithm illustrating the proposed method, where the numbers refer to the corresponding steps.

Step 7 (check for the existence of unique solutions). Identify all the variables with unique solution. If there are not variables with unique solution go to Step 2.

Step 8 (check for the existence of null variables). If any of these product variables has null value, identify the single variables having a null value.

Step 9 (identify variable signs). From the sign of the product variables derive the sign of their single variable terms.

Step 10 (generate and solve the system of logarithms of the absolute values of the known product variables). Take logarithms of the system of equations resulting from equating the absolute values of the known product variables to their corresponding values. This leads to a system of linear equations in the absolute values of the single variables involved.

Step 11 (check for the existence of unique solutions). If there are single variables with unique solution, identify them. Otherwise, go to Step 2.

Step 12 (check if nonunique product variables solutions can be learnt from the unique ones). Next, check which multiple solution variables can be learnt from the known single variables. Finally, go to Step 2.

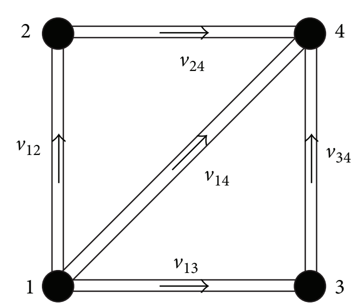

Figure 3: Example of a water supply system showing the nodes, pipes, and flow velocities.

7. Application Example 1: A Water Supply Network System

To illustrate the application of the proposed method, a water supply problem is considered. In Figure 3, a very simple example with four nodes and five pipes is shown. Though some of the illustrated steps could appear as trivial, in fact they are not so for more complex cases.

The equations that define this problem are

$$H_i = \frac{p_{ij}}{\gamma} + \frac{v_{ij}^2}{2g} + z_i; \quad \forall i \in \mathcal{N},\ \forall \ell_{ij} \in \mathcal{C}, \tag{38}$$

$$Q_i = \frac{\pi}{4}\sum_{\ell_{ij}} D_{ij}^2 v_{ij}; \quad \forall i \in \mathcal{N}, \tag{39}$$

$$H_i - H_j = \frac{\lambda_{ij} L_{ij}}{2g D_{ij}} V_{ij}^2; \quad \forall \ell_{ij} \in \mathcal{C}, \tag{40}$$

where $\mathcal{N}$ and $\mathcal{C}$ are the sets of nodes and pipes, respectively, z_i, H_i, and Q_i are the geometric height, the total height, and the flow of node i, and p_{ij}, v_{ij}, D_{ij}, and λ_{ij} are the pressure, the velocity, the diameter, and the friction coefficient associated with the pipe joining nodes i and j. For simplicity, in this example it is assumed that $z_i = 0$, $i = 1, 2, 3, 4$.

Equation (38) establishes that the total height is the sum of the piezometric height, the kinetic height, and the geometric height (the three terms in (38)). Equation (39) is the flow balance at node i; that is, the sum of flows entering the node and those leaving the node must be null. Finally, (40) establishes that the height loss is proportional to the velocity squared and inversely proportional to the diameter.

It is easy to see that system (38)–(40) is a multivariate polynomial system of equations of the form described in previous sections. As indicated, this is not a coincidence but the result of dealing with a physical problem.

System (38)–(40) for the particular case of our simple example becomes

$$0 = p_{12} + \frac{5}{98} v_{12}^2 - H_1, \tag{41}$$

$$0 = +p_{13} + \frac{5}{98} v_{13}^2 - H_1, \tag{42}$$

$$0 = +p_{14} + \frac{5}{98} v_{14}^2 - H_1, \tag{43}$$

$$0 = +p_{24} + \frac{5}{98} v_{24}^2 - H_2, \tag{44}$$

$$0 = +p_{34} + \frac{5}{98} v_{34}^2 - H_3, \tag{45}$$

$$\begin{aligned} 0 = &+\frac{\pi}{4} D_{12}^2 v_{12} \operatorname{sign}(v_{12}) + \frac{\pi}{4} D_{13}^2 v_{13} \operatorname{sign}(v_{13}) \\ &+ \frac{\pi}{4} D_{14}^2 v_{14} \operatorname{sign}(v_{14}) - Q_1 \operatorname{sign}(Q_1), \end{aligned} \tag{46}$$

$$\begin{aligned} 0 = &-\frac{\pi}{4} D_{12}^2 v_{12} \operatorname{sign}(v_{12}) + \frac{\pi}{4} D_{24}^2 v_{24} \operatorname{sign}(v_{24}) \\ &- Q_2 \operatorname{sign}(Q_2), \end{aligned} \tag{47}$$

$$\begin{aligned} 0 = &-\frac{\pi}{4} D_{13}^2 v_{13} \operatorname{sign}(v_{13}) + \frac{\pi}{4} D_{34}^2 v_{34} \operatorname{sign}(v_{34}) \\ &- Q_3 \operatorname{sign}(Q_3), \end{aligned} \tag{48}$$

$$\begin{aligned} 0 = &-\frac{\pi}{4} D_{14}^2 v_{14} \operatorname{sign}(v_{14}) - \frac{\pi}{4} D_{24}^2 v_{24} \operatorname{sign}(v_{24}) \\ &- \frac{\pi}{4} D_{34}^2 v_{34} \operatorname{sign}(v_{34}) - Q_4 \operatorname{sign}(Q_4), \end{aligned} \tag{49}$$

$$0 = +\frac{5}{98} D_{12}^{-1} v_{12}^2 \operatorname{sign}(v_{12}) - H_1 + H_2, \tag{50}$$

$$0 = +\frac{5}{98} D_{13}^{-1} v_{13}^2 \operatorname{sign}(v_{13}) - H_1 + H_3, \tag{51}$$

$$0 = +\frac{5}{98} D_{14}^{-1} v_{14}^2 \operatorname{sign}(v_{14}) - H_1 + H_4, \tag{52}$$

$$0 = +\frac{5}{98} D_{24}^{-1} v_{24}^2 \operatorname{sign}(v_{24}) - H_2 + H_4, \tag{53}$$

$$0 = +\frac{5}{98} D_{34}^{-1} v_{34}^2 \operatorname{sign}(v_{34}) - H_3 + H_4. \tag{54}$$

Note that (50) to (54) require the use of sign variables, which are not dealt with by the existing packages for solving systems of polynomial equations.

In this example, we measure the total heights H_1 and H_2 and the flows Q_1 and Q_2 and we use piezometers at the beginning of all pipes to measure the pressures p_{ij}; $\ell_{ij} = 1, 2, 3, 4, 5$.

Next, the use of the algorithm by a detailed explanation of all its steps is illustrated.

Input. The data of this problem are

$$\begin{aligned} \overline{V}_0 &= \{p_{12}, p_{13}, p_{14}, p_{24}, p_{34}, H_1, H_2, Q_1, Q_2, \operatorname{sign}(Q_1), \\ &\quad \operatorname{sign}(Q_2)\}, \\ \overline{V}_0^0 & \\ &= \{11.38, 12.65, 10, 10.16, 9.73, 20, 12, 33.24, 7.89, 1, \\ &\quad -1\}. \end{aligned} \tag{55}$$

So, store $\overline{V}_0$ and $\overline{V}_0^0$ into $\mathcal{O}$ and $\mathcal{O}^0$, respectively.

Step 1 (build the initial system of equations). The initial matrices B, z, and D are immediately derived from (41) to (54) to get

$$\begin{pmatrix}
1&0&0&0&0&0.05&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0.05&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0.05&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0.05&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0.05&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0.79&0.79&0.79&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-0.79&0&0&0.79&0&0&0&0&0&0&0&0&0&0&0&-1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-0.79&0&0&0.79&0&0&0&0&0&0&0&0&0&0&0&-1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&-0.79&-0.79&-0.79&0&0&0&0&0&0&0&0&0&0&0&0&-1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0.05&0&0&0&0&-1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0.05&0&0&0&-1&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0.05&0&0&-1&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0.05&0&0&-1&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0.05&0&0&-1&1&0&0&0&0
\end{pmatrix}
\begin{pmatrix}
p_{12}\\ p_{13}\\ p_{14}\\ p_{24}\\ p_{34}\\ v_{12}^2\\ v_{13}^2\\ v_{14}^2\\ v_{24}^2\\ v_{34}^2\\
D_{12}^{2} v_{12} \operatorname{sign}(v_{12})\\
D_{13}^{2} v_{13} \operatorname{sign}(v_{13})\\
D_{14}^{2} v_{14} \operatorname{sign}(v_{14})\\
D_{24}^{2} v_{24} \operatorname{sign}(v_{24})\\
D_{34}^{2} v_{34} \operatorname{sign}(v_{34})\\
D_{12}^{-1} v_{12}^2 \operatorname{sign}(v_{12})\\
D_{13}^{-1} v_{13}^2 \operatorname{sign}(v_{13})\\
D_{14}^{-1} v_{14}^2 \operatorname{sign}(v_{14})\\
D_{24}^{-1} v_{24}^2 \operatorname{sign}(v_{24})\\
D_{34}^{-1} v_{34}^2 \operatorname{sign}(v_{34})\\
H_1\\ H_2\\ H_3\\ H_4\\
Q_1 \operatorname{sign}(Q_1)\\ Q_2 \operatorname{sign}(Q_2)\\ Q_3 \operatorname{sign}(Q_3)\\ Q_4 \operatorname{sign}(Q_4)
\end{pmatrix}
= \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{pmatrix}. \tag{56}$$

Step 2 (check whether or not known variables are available). Since there are new known variables, go to Step 3.

Step 3 (replace known variables by their values and modify the system of equations). Replace all known variables in matrix z by their values and move them to matrix B to get

$$\left(\begin{array}{cccccccccccccccccccccccccccc}
11.38 & 0 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -20.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 12.65 & 0 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -20.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 10.00 & 0 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -20.00 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 10.16 & 0 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -12.00 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 9.73 & 0 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.79 & 0.79 & 0.79 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -33.24 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.79 & 0 & 0 & 0.79 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7.89 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.79 & 0 & 0 & 0.79 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.79 & -0.79 & -0.79 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0 & -20.00 & 12.00 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0 & -20.00 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0 & 0 & -20.00 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0 & 0 & -12.00 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0
\end{array}\right)
\begin{pmatrix}
1 \\ 1 \\ 1 \\ 1 \\ 1 \\ v_{12}^2 \\ v_{13}^2 \\ v_{14}^2 \\ v_{24}^2 \\ v_{34}^2 \\
D_{12}^2 v_{12} \operatorname{sign}(v_{12}) \\ D_{13}^2 v_{13} \operatorname{sign}(v_{13}) \\ D_{14}^2 v_{14} \operatorname{sign}(v_{14}) \\ D_{24}^2 v_{24} \operatorname{sign}(v_{24}) \\ D_{34}^2 v_{34} \operatorname{sign}(v_{34}) \\
D_{12}^{-1} v_{12}^2 \operatorname{sign}(v_{12}) \\ D_{13}^{-1} v_{13}^2 \operatorname{sign}(v_{13}) \\ D_{14}^{-1} v_{14}^2 \operatorname{sign}(v_{14}) \\ D_{24}^{-1} v_{24}^2 \operatorname{sign}(v_{24}) \\ D_{34}^{-1} v_{34}^2 \operatorname{sign}(v_{34}) \\
1 \\ 1 \\ H_3 \\ H_4 \\ 1 \\ 1 \\ Q_3 \operatorname{sign}(Q_3) \\ Q_4 \operatorname{sign}(Q_4)
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{57}$$

Step 4 (test for known product variables). Since there are new known product variables (those showing one in the column matrix in (58)) subtract the corresponding columns from the independent term matrix D and remove the columns from B and the variables from z to get

$$\begin{pmatrix}
0.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.79 & 0.79 & 0.79 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.79 & 0 & 0 & 0.79 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.79 & 0 & 0 & 0.79 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.79 & -0.79 & -0.79 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & -1 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
v_{12}^2 \\ v_{13}^2 \\ v_{14}^2 \\ v_{24}^2 \\ v_{34}^2 \\
D_{12}^2 v_{12} \operatorname{sign}(v_{12}) \\
D_{13}^2 v_{13} \operatorname{sign}(v_{13}) \\
D_{14}^2 v_{14} \operatorname{sign}(v_{14}) \\
D_{24}^2 v_{24} \operatorname{sign}(v_{24}) \\
D_{34}^2 v_{34} \operatorname{sign}(v_{34}) \\
D_{12}^{-1} v_{12}^2 \operatorname{sign}(v_{12}) \\
D_{13}^{-1} v_{13}^2 \operatorname{sign}(v_{13}) \\
D_{14}^{-1} v_{14}^2 \operatorname{sign}(v_{14}) \\
D_{24}^{-1} v_{24}^2 \operatorname{sign}(v_{24}) \\
D_{34}^{-1} v_{34}^2 \operatorname{sign}(v_{34}) \\
H_3 \\ H_4 \\
Q_3 \operatorname{sign}(Q_3) \\
Q_4 \operatorname{sign}(Q_4)
\end{pmatrix}
= \begin{pmatrix}
8.622 \\ 7.347 \\ 10.000 \\ 1.837 \\ -9.735 \\ 33.239 \\ -7.886 \\ 0 \\ 0 \\ 8.000 \\ 20.000 \\ 20.000 \\ 12.000 \\ 0
\end{pmatrix}. \tag{58}$$

Step 5 (check for replicated product variables). Since there are not replicated product variables, go to Step 6.

Step 6 (solve the system of linear equations in product variables). Solving the system of (58), we obtain the following general solution:

$$\begin{pmatrix} v_{12}^2 \\ v_{13}^2 \\ v_{14}^2 \\ v_{24}^2 \\ v_{34}^2 \\ D_{12}^2 v_{12}\operatorname{sign}(v_{12}) \\ D_{13}^2 v_{13}\operatorname{sign}(v_{13}) \\ D_{14}^2 v_{14}\operatorname{sign}(v_{14}) \\ D_{12}^{-1} v_{12}^2\operatorname{sign}(v_{12}) \\ D_{13}^{-1} v_{13}^2\operatorname{sign}(v_{13}) \\ D_{14}^{-1} v_{14}^2\operatorname{sign}(v_{14}) \\ D_{24}^{-1} v_{24}^2\operatorname{sign}(v_{24}) \\ D_{34}^{-1} v_{34}^2\operatorname{sign}(v_{34}) \\ Q_3\operatorname{sign}(Q_3) \end{pmatrix} = \begin{pmatrix} 169.000 \\ 144.000 \\ 196.000 \\ 36.000 \\ -190.800 \\ 10.041 \\ 32.279 \\ 0 \\ 156.800 \\ 392.000 \\ 392.000 \\ 235.200 \\ 0 \\ -25.352 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 19.600 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1.273 \\ -1 & -1 & 0 & 0 & -1.273 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -19.600 & 0 & 0 \\ 0 & 0 & 0 & -19.600 & 0 \\ 0 & 0 & 0 & -19.600 & 0 \\ 0 & 0 & 19.60 & -19.600 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} D_{24}^2 v_{24}\operatorname{sign}(v_{24}) \\ D_{34}^2 v_{34}\operatorname{sign}(v_{34}) \\ H_3 \\ H_4 \\ Q_4\operatorname{sign}(Q_4) \end{pmatrix}. \quad (59)$$

Step 7 (check for the existence of unique solutions). Since null rows in the null space matrix in (59) are found, unique solutions for the set of variables are

$$\{v_{12}^2 \ \ v_{13}^2 \ \ v_{14}^2 \ \ v_{24}^2 \ \ D_{12}^2 v_{12}\operatorname{sign}(v_{12})\}. \quad (60)$$

More precisely, we have

$$\begin{pmatrix} v_{12}^2 \\ v_{13}^2 \\ v_{14}^2 \\ v_{24}^2 \\ D_{12}^{-1} v_{12}^2\operatorname{sign}(v_{12}) \end{pmatrix} = \begin{pmatrix} 169.00 \\ 144.00 \\ 196.00 \\ 36.00 \\ 156.80 \end{pmatrix}. \quad (61)$$

Step 8 (check for the existence of null variables). Since none of these values is null, there are no null variables.

Step 9 (identify variable signs). From the signs of these product variables, we derive that $\operatorname{sign}(v_{12}) = 1$.

Step 10 (generate and solve the system of logarithms of the absolute values of the known product variables). Taking logarithms in (61) the following linear system of equations is obtained:

$$\log\begin{pmatrix}169.00\\144.00\\196.00\\36.00\\156.80\end{pmatrix}=\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}\log\begin{pmatrix}p_{12}\\p_{13}\\p_{14}\\p_{24}\\p_{34}\\v_{12}\\v_{13}\\v_{14}\\v_{24}\\v_{34}\\D_{12}\\D_{13}\\D_{14}\\D_{24}\\D_{34}\\H_1\\H_2\\H_3\\H_4\\Q_1\\Q_2\\Q_3\\Q_4\end{pmatrix} \tag{62}$$

with general solution given by

$$\log\begin{pmatrix}p_{12}\\p_{13}\\p_{14}\\p_{24}\\p_{34}\\v_{12}\\v_{13}\\v_{14}\\v_{24}\\v_{34}\\D_{12}\\D_{13}\\D_{14}\\D_{24}\\D_{34}\\H_1\\H_2\\H_3\\H_4\\Q_1\\Q_2\\Q_3\\Q_4\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\2.565\\2.485\\2.639\\1.792\\0\\0.075\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}$$

$$+\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

$$\cdot \log \begin{pmatrix} p_{12} \\ p_{13} \\ p_{14} \\ p_{24} \\ p_{34} \\ v_{34} \\ D_{13} \\ D_{14} \\ D_{24} \\ D_{34} \\ H_1 \\ H_2 \\ H_3 \\ H_4 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix}. \tag{63}$$

Step 11 (check for the existence of unique solutions). From the above solution, the set of variables with unique solution is identified (it corresponds to null rows of the null space matrix), which is

$$\begin{pmatrix} v_{12} \\ v_{13} \\ v_{14} \\ v_{24} \\ D_{12} \end{pmatrix} = \begin{pmatrix} 13.000 \\ 12.000 \\ 14.000 \\ 6.000 \\ 1.078 \end{pmatrix}. \tag{64}$$

Step 12 (check if nonunique product variables solutions can be learnt from the unique ones). Next, check if some of the multiple solution variables in (59)

$$\begin{pmatrix} v_{34}^2 \\ D_{12}^2 v_{12} \operatorname{sign}(v_{12}) \\ D_{13}^2 v_{13} \operatorname{sign}(v_{13}) \\ D_{14}^2 v_{14} \operatorname{sign}(v_{14}) \\ D_{24}^2 v_{24} \operatorname{sign}(v_{24}) \\ D_{34}^2 v_{34} \operatorname{sign}(v_{34}) \\ D_{13}^{-1} v_{13}^2 \operatorname{sign}(v_{13}) \\ D_{14}^{-1} v_{14}^2 \operatorname{sign}(v_{14}) \\ D_{24}^{-1} v_{24}^2 \operatorname{sign}(v_{24}) \\ D_{34}^{-1} v_{34}^2 \operatorname{sign}(v_{34}) \\ H_3 \\ H_4 \\ Q_3 \operatorname{sign}(Q_3) \\ Q_4 \operatorname{sign}(Q_4) \end{pmatrix} \tag{65}$$

can be determined from the values in (64). It is found that the first five variables can be determined, but not the other four.

To this end, the following product of matrices is calculated:

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

$$= \begin{pmatrix} 0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&2&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&2&0&0&0&-1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{pmatrix}, \tag{66}$$

where the first matrix corresponds to variables in (59) and the second one is the null space matrix in (63). Since the second row from the last matrix is null, variable $D_{12}^2 v_{12}$ can be obtained from (64).

Next, go to Step 2 and iterate the process until no new variables are obtained. After 6 iterations, full observability is obtained (all variables can be calculated from the values of the data variables). From the initial data,

$$\begin{aligned} p_{12} &= 11.38; \\ p_{13} &= 12.65; \\ p_{14} &= 10; \\ p_{24} &= 10.16; \\ p_{34} &= 9.73; \\ H_1 &= 20; \\ H_2 &= 12; \\ Q_1 &= 33.24; \\ Q_2 &= 7.89; \\ \operatorname{sign}(Q_1) &= 1; \\ \operatorname{sign}(Q_2) &= -1, \end{aligned} \tag{67}$$

and the remaining variable values become

$$\begin{aligned} \operatorname{sign}(v_{12}) &= 1; \\ v_{12} &= 13; \\ v_{13} &= 12; \\ v_{14} &= 14; \\ v_{24} &= 6; \\ D_{12} &= 1.08; \\ \operatorname{sign}(v_{24}) &= 1; \\ D_{24} &= 0.92; \\ \operatorname{sign}(v_{14}) &= 1; \\ H_4 &= 10; \\ D_{14} &= 1; \\ \operatorname{sign}(v_{13}) &= 1; \\ D_{13} &= 1.05; \\ \operatorname{sign}(v_{34}) &= 1; \\ v_{34} &= 8; \\ H_3 &= 13; \\ D_{34} &= 1.09; \\ Q_3 &= 2.94; \\ Q_4 &= 22.41; \\ \operatorname{sign}(Q_3) &= -1; \\ \operatorname{sign}(Q_4) &= -1. \end{aligned} \tag{68}$$

Table 1 summarizes the data and what is learnt in each recursive step and operation and Table 2 shows other interesting observability cases.

A similar observability problem from the field of electrical engineering can be seen in [28].

TABLE 1: A summary of the data and what is learnt in each recursive step and operation.

Recursive step	Variables learnt in Operation 2	Sign functions learnt from observed and unobserved variables	Variables learnt in Operation 3 from observed and unobserved variables
Data: $p_{12}, p_{13}, p_{14}, p_{24}, p_{34}, H_1, H_2, Q_1, Q_2$			
1	$v_{12}^2, v_{13}^2, v_{14}^2, v_{24}^2, D_{12}^{-1} v_{12}^2 \mathrm{Sign}(v_{12})$	$\mathrm{Sign}(v_{12})$	$v_{12}, v_{13}, v_{14}, v_{24}, D_{12}$
2	$\mathrm{Sign}(v_{24}) D_{24}^2$	$\mathrm{Sign}(v_{24})$	D_{24}
3	$\mathrm{Sign}(v_{14}) D_{14}^{-1}, H_4$	$\mathrm{Sign}(v_{14})$	D_{14}, H_4
4	$\mathrm{Sign}(v_{13}) D_{13}^2$	$\mathrm{Sign}(v_{13})$	D_{13}
5	$v_{34}^2, D_{34}^{-1} v_{34}^2 \mathrm{Sign}(v_{34})$	$\mathrm{Sign}(v_{34})$	v_{34}, D_{34}, H_3
6	Q_3, Q_4	—	—

TABLE 2: Some examples of total or partial observability in the water supply problem.

Known data	Observable data	Full observability
H_i and D_{ij}	Q_i, p_{ij}, and v_{ij}	Yes
H_i and p_{ij}	Q_i, v_{ij}, and D_{ij}	Yes
H_i and v_{ij}	Q_i, p_{ij}, and D_{ij}	Yes
Q_i and D_{ij}	—	No
Q_i, p_{ij}	—	No
Q_i and v_{ij}	—	No

8. Application Example 1: Calculus of Structures—The Nationale-Nederlanden Building in Downtown Prague

In this section, a significative example of calculus of structures is given: the Nationale-Nederlanden building in downtown Prague.

Structural system identification (SSI) or inverse analysis can be defined as the process of updating or calibrating a simplistic physically based model of a structure (e.g., finite element model) based on actual structure information (health monitoring). This topic has received significant attention in the last decades (see, e.g., [29, 30]), as an accurate model of a structure is always needed to evaluate and predict in-service structural behavior and to support operational and maintenance decisions. However, sources of uncertainty, as material or mechanical properties or damage, make such a task very difficult. To illustrate the applicability of the proposed method in the SSI field, a simplified (and still quite complex) two-dimensional model of the Nationale-Nederlanden building (Figure 4) in Prague, Czech Republic, is analyzed. This building, also known as the Dancing House, was designed by Frank Gehry in cooperation with Vlado Milunic in 1992 and it was completed in 1996.

To analyze this structure, the stiffness matrix method can be used, which provides a linear system of equations relating node forces and moments with node displacements and rotations. The system is linear only if the mechanical and geometric characteristics of the beam and column elements are known. However, in our case, they are unknown, and then a polynomial system of equations (see [31] for details) of a type similar to (22) and (41)–(54) is obtained.

FIGURE 4: Nationale-Nederlanden building in downtown Prague (Czech Republic). Picture captured from http://en.wikipedia.org/wiki/File:Case_danzanti.jpg.

The 8-story building is modeled by 135 nodes and 155 bars as presented in Figure 5. The structure is composed of a set of 6 (from I to VI) different sections. Young's modulus, E_i, area, A_i, and inertia, I_i, of each section i are described in Table 3. The use of the stiffness matrix method (see [32]) results in a system of 405 equations. As indicated, the problem becomes nonlinear because we assume that the bar mechanical properties (E, A, I) and the node deflections (u, v, w) are unknown producing the following nonlinear products of variables $E_i A_i u_j$, $E_i A_i v_j$, $E_i I_i u_j$, $E_i I_i v_j$, and $E_i I_i w_j$.

Table 3: Characteristics of the elements of the building.

Section	Elements	E	A	I
(I) Bottom columns	1 to 4, 29 to 36, 53 to 60, 77 to 84, 101 to 104	E_1	A_1	I_1
(II) Cantilever	21 to 28	E_2	A_2	I_2
(III) Outer columns	5 to 20, 85 to 100	E_3	A_3	I_3
(IV) Interior columns	37 to 52, 61 to 76	E_4	A_4	I_4
(V) Inclined intermediate columns	105 to 106	E_5	A_5	I_5
(VI) Floor slab	107 to 155	E_6	A_6	I_6

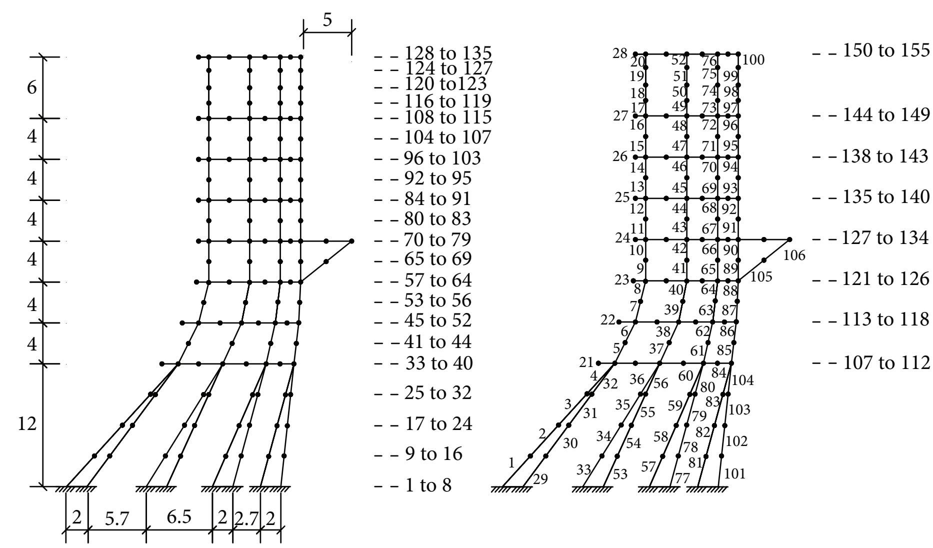

Figure 5: Model with 135 nodes and 155 bars used to reproduce the Nationale-Nederlanden building. Dimensions are given in m.

The aim of the example is to determine the actual axial, EA, and bending, EI, stiffness of the 6 sections based on a subset of measured inputs. These inputs can be obtained by measuring deflections and/or rotations in the actual structure by topography and clinometers. To analyze the observability of the pursued parameters, the subset of deflections and rotations presented in Figure 6 are assumed to be measured. It is worth mentioning that knowledge of the actual mechanical properties of the structure guarantees a more accurate simulation of the structure with the consequent assessment of structural safety.

The observable variables (including u, v, and w and node reactions), the observable axial stiffness EA, and the observable flexural stiffness, EI, obtained throughout the recursive process, are presented in Table 4. This table shows that after six recursive steps all parameters of the structure (341) become observable. The steps of the proposed algorithm cannot be shown due to the high size of the matrices implied. However, since the stiffness matrix is of the type in expression (1), the unknowns are E_i and A_i, and the used algorithm has been illustrated with the two previous examples, the reader can easily imagine how the results were obtained.

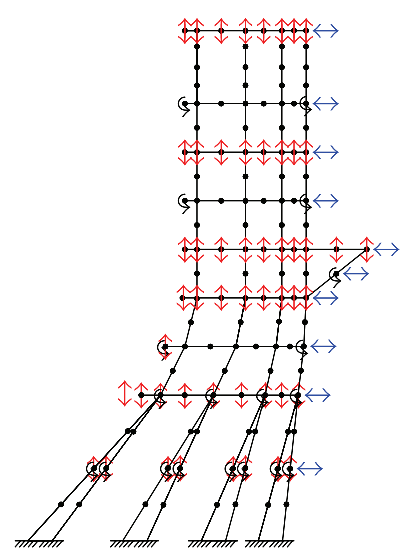

Figure 6: Subset of deflections and rotations used to obtain the mechanical properties of the model elements.

Table 4: Observable variables throughout the recursive process.

Rec. step	Observations	Observed EA	Observed EI
1	21	—	E_2I_2, E_6I_6
2	94	E_3A_3, E_4A_4	E_2I_2, E_3I_3, E_6I_6
3	156	E_3A_3, E_4A_4	$E_2I_2, E_3I_3, E_4I_4, E_5I_5, E_6I_6$
4	272	$E_3A_3, E_4A_4, E_5A_5, E_6A_6$	$E_1I_1, E_2I_2, E_3I_3, E_4I_4, E_5I_5, E_6I_6$
5	291	$E_1A_1, E_3A_3, E_4A_4, E_5A_5, E_6A_6$	$E_1I_1, E_2I_2, E_3I_3, E_4I_4, E_5I_5, E_6I_6$
6	331	$E_1A_1, E_2A_2, E_3A_3, E_4A_4, E_5A_5, E_6A_6$	$E_1I_1, E_2I_2, E_3I_3, E_4I_4, E_5I_5, E_6I_6$

Table 5: Comparison among different software packages for solving polynomial systems of equations. The symbol — means that the program was not able to solve it.

Example	Number of equations	Proposed method	Maple (PolynomialSystem)	Mathematica (Solve)	Bertini software
Water supply Example 1	14	4.88 sec	18.33 sec	13.20 sec	>2 hours
Structure example	30	7.20 sec	—	—	>8 hours
Dancing House	405	12.50 sec	—	—	>8 hours

This example supports the applicability of the proposed method as a powerful tool in the SSI field, even for the case of complex structures.

9. A Comparison of Different Methods

In this section, the proposed method is compared with some standard methods for solving polynomial systems of equations. In particular, the Maple function "PolynomialSystem," Mathematica function "Solve," and Bertini's software [33, 34] have been used.

Maple, Mathematica, and Bertini provide the complete set of possible solutions of the polynomial equation systems. Given the nature of these systems, this set likely contains a large number of possible solutions. It is noted that only some few combinations have physical meaning.

When attempting to solve systems of 20 polynomial equations, Maple, Mathematica, and Bertini were unable to provide all the possible combinations in a reasonable time (after several hours of calculation, the programs were aborted). Moreover, dealing with smaller system of equations allowed the identification of another important handicap. That is, when the data were subject to numerical errors, the system of equations had no solutions due to the presence of redundancies.

Contrary, the proposed method, which works numerically, provided the solutions in a very reasonable time (seconds), even for large size examples. This is due to the fact that the proposed method avoids redundancies, because once some unknowns have been determined they are replaced in the other equations and when they are degenerated they are ignored.

In Table 5, the CPU times required for some examples using the proposed methods and the indicated functions are shown.

Finally, it is worthwhile mentioning that since the sparsity of the set of equations is related to the degree of structural redundancy, the larger the redundancy the larger will be the difficulty in solving the problem.

10. Conclusions

The main conclusions that can be derived from this paper are the following.

(1) The provided original method permits us to obtain in a reasonable time the unique solutions of multivariate polynomial systems of equations whose coefficient matrix has a proper null space so that the values of a subset of monomials can be obtained.

(2) It has been justified why these systems of equations arise in physical and engineering problems. In fact, no other system can be expected if physically valid equations are dealt with.

(3) The observability techniques can be satisfactorily and efficiently applied to solve important engineering problems as it has been demonstrated with the applications in this paper.

(4) The observability techniques permit the identification of the structural systems parameters even though the variables involved are not linearly related.

(5) Before applying the techniques proposed in this paper, the traditional mathematical statement of the problems needs to be transformed in such a way that all monomials appear separated.

(6) If the solution of the system of equations is not complete (not all variables have a unique solution), some relations are obtained that allow us to determine which variables must be observed to have full or a more complete observability.

(7) Though some packages already include this possibility, it would be good for polynomial systems

of equations packages to include options limiting solutions to be real and nonnegative, the possibility of determining absolute values of variables, and the use of sign variables. In addition faster methods are required to be competitive with ad hoc or tailor made methods as the one proposed in this paper. Finally, allowing for numerical errors in the polynomial equations is unavoidable.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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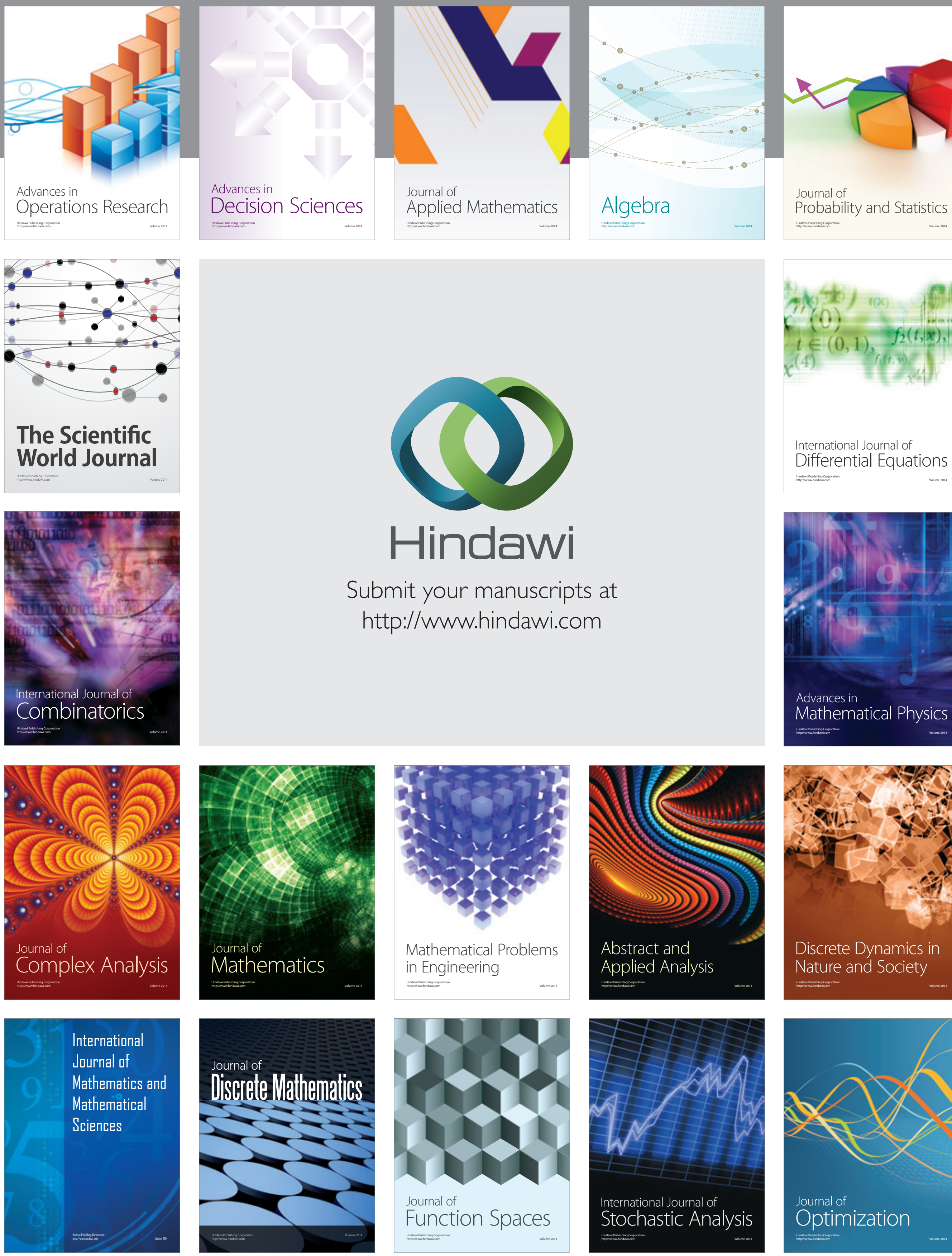
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